Structural Credit Model with Time-Varying Default Barriers - Working Note

Andrea Bortolini

Motivation and Idea

Merton (1974) is maturity-blind: ceteris paribus, two firms with identical leverage but different debt schedules get the same default probability. Black and Cox (1976) introduced exponential default barriers, Reiner and Rubinstein (1991) provided closed-form barrier-option pricing, and He and Xiong (2012) showed how rollover risk endogenously shifts the default boundary.

In this note, I build on these ideas to explore a model in which the time-varying default barrier is parameterized using 10-K maturity-ladder disclosures, with scaling parameters chosen illustratively. Two scenarios are possible: *decay* (the barrier declines as near-term debt matures off the balance sheet) and *stress* (refinancing pressure on long-term debt pushes the barrier upward). After scraping maturity-ladder data from several firms' 10-K filings, I chose American Airlines (AAL) and Carnival Corporation (CCL) as two illustrative examples with notably different maturity profiles: AAL has a front-loaded maturity schedule, which maps naturally into the runoff/decay specification, while CCL has a back-loaded schedule with high rollover dependence, which makes it a useful illustration of the stress specification.

Setup

Under the risk-neutral measure $\mathbb{Q}$, asset value follows $dV/V = (r-y)\,dt + \sigma\,dW$, with risk-free rate r and payout yield y (the continuous rate at which cash leaves the firm via dividends or buybacks). Default occurs at first passage of an exponential barrier:

$$B_i(t) = B_0\,e^{-\eta_i t}, \qquad \eta_i > 0 \text{ (decay)}, \quad \eta_i = 0 \text{ (Black–Cox)}, \quad \eta_i < 0 \text{ (stress)}. \tag{1}$$

The barrier $B_i(t)$ is an early-distress threshold (the asset level below which the firm cannot service near-term obligations), not the full face value F. Thus the model has two equity-loss channels: first-passage default prior to T, and terminal insolvency when $V_T \le F$. In particular $B_i(T) < V_T \le F$ is admissible: the firm avoids early default but equity is still wiped out at maturity. Defining $Y_t = \ln(V_t/B_i(t))$ reduces the moving-barrier problem to first passage of arithmetic Brownian motion at zero, with drift $m = r - y - \frac{1}{2}\sigma^2 + \eta_i$, which is what makes the reflection-based closed form feasible.

Equity formula. Under continuous monitoring, zero recovery at first passage, and in the regime $V_0 > B_0$ and $F > B_0\,e^{-\eta_i T}$, equity can be written as a down-and-out call:

$$E_0 = C_{BS}(V_0, F) - \Big(\frac{V_0}{B_0}\Big)^{-2\beta} C_{BS}\Big(\frac{B_0^2}{V_0},\, F\Big), \qquad \beta = \frac{r - y - \frac{1}{2}\sigma^2 + \eta_i}{\sigma^2}. \tag{2}$$

The slope η_i enters only through the reflection exponent β, not through the strike. After writing the payoff as $(V_T - F)^+ 1_{\{\tau > T\}}$ and transforming to log-distance from the exponential barrier, the

$\eta_i T$ terms cancel in both the direct and reflected Gaussian integrals; the image term reflects the spot, not the strike. This has been verified analytically (exact reduction to Black–Cox at $\eta_i = 0$) and numerically (Brownian-bridge-corrected Monte Carlo across a grid of η_i, σ, and V_0/B_0). I have a short appendix with the derivation and the analytic equity delta if that would be useful.

KMV inversion. The barrier parameters (B_0, η_i) are fixed from disclosures before estimating asset value. The unknowns (V_0, σ) are backed out from market capitalization E_0^{mkt} and equity volatility σ_E by solving: (1) the equity formula (2) must match E_0^{mkt}, and (2) by Itô's lemma, $\sigma_E\, E_0^{\text{mkt}} = (\partial E_0/\partial V_0)\, \sigma\, V_0$, where $\partial E_0/\partial V_0$ is the equity delta, i.e. the sensitivity of equity value to asset value. This delta has a closed-form three-term expression that corrects the standard Merton delta for the barrier. In my numerical experiments, Newton's method converges quickly from standard KMV-style initial guesses.

Calibration of Level and Slope

The initial barrier is $B_0 = \lambda\, S_i$, where S_i is debt due within one year and $\lambda = 1.25$ (illustrative). The signed barrier drift decomposes as

$$\eta_i = \underbrace{\kappa\, a_i}_{\text{decay}} - \underbrace{\varphi\, \rho_i\, z_i}_{\text{stress}}, \tag{3}$$

where $a_i = (M_1 + M_2)/\sum_{k=1}^{5} M_k$ is the front-loading share (M_k = debt maturing in year k from the 10-K maturity ladder), $\rho_i = L_i/(S_i + L_i)$ is the long-term debt share, $z_i \in \{0, 1\}$ is a reduced-form regime indicator, $\kappa = 0.50$, and $\varphi = 0.25$. The barrier rises under stress when $\rho_i/a_i > \kappa/\varphi$: firms whose capital structure implies heavier future refinancing needs and whose near-term schedule is light. Here z_i is assigned heuristically from the maturity profile to illustrate how the same pricing framework can accommodate both runoff and refinancing-stress cases.

Note: the scaling parameters λ, κ, and φ are chosen arbitrarily for illustrative purposes. A next step is to calibrate them more precisely, for instance by fitting to observed CDS spreads or historical default frequencies.

Invariance result. The decay channel's contribution to distance-to-default is $\Delta\text{DD}_{\text{decay}} = \kappa\, a_i/\sigma$. Within this parametrization, it is independent of the barrier level B_0 and of λ; it depends only on the maturity-ladder share a_i, asset volatility, and the chosen decay loading κ. Economically, the decay channel adds κa_i to the *drift* of the log-distance process Y. Because drift affects the rate of change of DD rather than its level, it is independent of where the barrier sits initially.

Illustration: AAL and CCL

AAL is assigned to *decay* ($z = 0$, front-loaded ladder), CCL to *stress* ($z = 1$, high rollover dependence). The numbers below should be read as a worked illustration of the mapping from maturity structure to implied DD, not as a disciplined empirical calibration. E^{mkt} is price × shares outstanding; σ_E is estimated from historical daily equity returns. Debt inputs are taken or reconstructed from FY2025 10-K disclosures; some maturity-ladder values are approximate principal estimates where reported payment schedules include estimated interest.

AAL has a front-loaded ladder ($a = 0.47$) and low asset volatility ($\sigma = 11.3\%$). Under decay, the declining barrier adds $+2.06$ to DD relative to Black–Cox, exactly $\kappa a/\sigma$ as predicted by the

	AAL (decay)	**CCL** (stress)
S_i / L_i / F_i ($M)	3,753 / 25,254 / 29,007	1,900 / 24,037 / 25,937
E^{mkt} ($M) / σ_E	7,000 / 55%	35,000 / 50%
5-yr ladder ($B)	3.8 / 3.5 / 3.0 / 2.8 / 2.5	1.9 / 2.5 / 4.0 / 4.0 / 3.0
B_0, a_i/ρ_i, η_i	4,691, 0.47/0.87, +0.233	2,375, 0.29/0.93, −0.089
V_0 / σ (inverted)	34,755 / 11.3%	59,841 / 29.3%
DD: Black–Cox ($\eta = 0$)	18.35	11.22
DD: assigned scenario	20.41	10.92
ΔDD vs. Black–Cox	+2.06	−0.30

invariance result; the low volatility amplifies the effect. *CCL* has a back-loaded ladder ($a = 0.29$) and high rollover dependence ($\rho = 0.93$): a large share of its debt will need to be refinanced over time. Under the stress specification, $\eta < 0$, so the barrier rises and DD falls relative to Black–Cox. While the exact magnitude depends on the treatment of current maturities, interest-inclusive payment schedules, and equity volatility, the qualitative stress direction is robust. That interpretation is qualitatively consistent with CCL's still-heavy forward maturity profile as of FY2025, even after the refinancing activity completed in 2025, though the note does not yet test whether the model improves explanatory or predictive performance. Varying λ shifts DD levels but preserves all slope effects identically.

Open Questions

Q1. Is it economically reasonable to distinguish the early-distress barrier $B(t)$ from terminal debt face value F, or would it make sense imposing $B(T) = F$ for consistency?

Q2. What empirical tests would best assess performance relative to standard KMV/Merton?

Q3. Are there papers on firm-level time-varying barriers or disclosure-based calibration that you think would be especially important to look at?

I would greatly appreciate any feedback on the economic framing, the calibration, or directions worth pursuing.

References. Black & Cox (1976), *J. Finance* 31(2). · He & Xiong (2012), *J. Finance* 67(2). · Merton (1974), *J. Finance* 29(2). · Reiner & Rubinstein (1991), *Risk* 4(8).